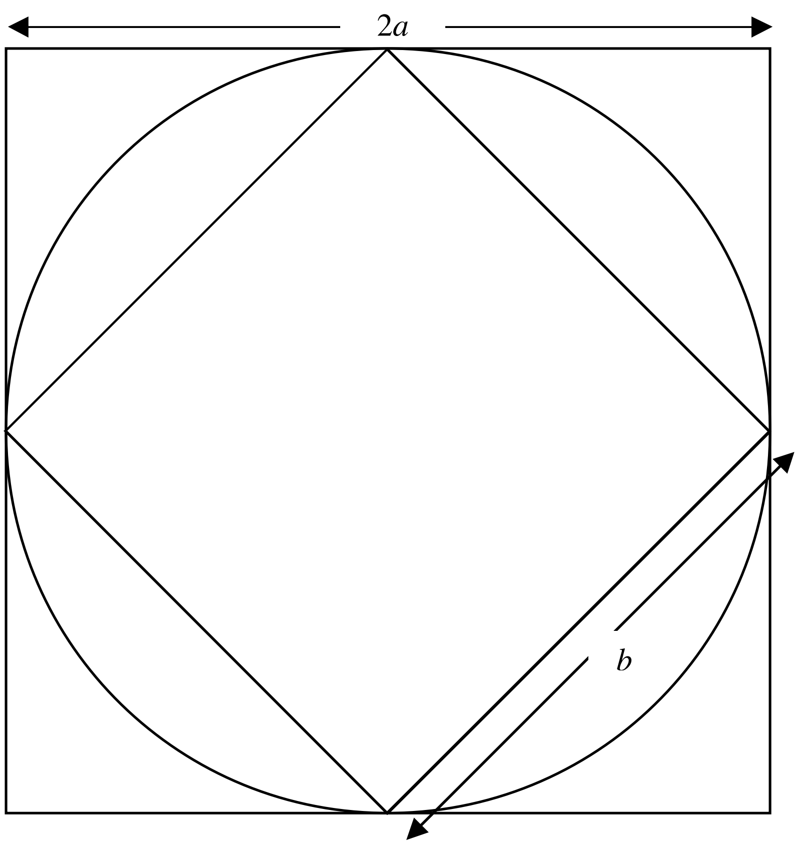
Circles and Squares

This problem gives you the chance to:
- interpret a mathematical diagram
- show your understanding of area
- use algebra in context

The diagram shows a square with a circle inside it, and a smaller square inside the circle.
The large square has sides of length $2a$ and the smaller square has sides of length b.



1. Explain why the diagram shows that $b^2 <$ area of the circle $< 4a^2$.

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2. By finding b in terms of a, show that $2a^2$ < area of the circle < $4a^2$.
Show your method clearly.

6

Circles and Squares — Rubric

The core elements of performance required by this task are: • interpret a mathematical diagram • show your understanding of area • use algebra in context Based on these, credit for specific aspects of performance should be assigned as follows	points	section points
1. Calculates the areas of the smaller and large squares: $\mathbf{b^2}$ and $\mathbf{4a^2}$ States that the area of circle lies between these two.	2 x 1 1	3
2. Uses the Pythagorean Rule to show that $b = \sqrt{2}a$ or $b^2 = 2a^2$ **or** Shows that the area of the small square is 2 x 1/2 x 2a x a $= 2a^2$ Finds that the area of the smaller square $= b^2 = 2a^2$ Accept alternative correct methods States that the area of circle lies between $2a^2$ and $4a^2$	3	3
Total Points		**6**

Circles and Squares

Work the task and look at the rubric. How would you like students to connect the diagram to the algebraic expressions in the task? What are the geometrical concepts that the student needs to make to bridge the two representations?

What do you want in a good mathematical argument? How do students learn the logic of making a convincing argument? How do you help them learn to distinguish between stating the conclusion and giving supporting evidence to back up the conclusion?
How do students learn to make sense of diagrams and make the connection between geometrical representations and algebraic representations?

Look at student work for part one of the task.

- How many of your students marked up their diagram, maybe showing the radius, to help them make sense and connect the diagram to the algebraic expressions? What are some of the ways that you might expect students to make use of the diagram?
- How many of your students just restated how the diagram was made?
- How many of your students could state that b^2 was the same as the area of the small square? Did any of them give a reason about why this was true or how they calculated area?
- How many of your students stated that the $4a^2$ was the same as the area of the large square?
- Did any of your students quantify the area of the circle or did they just rely on the logic of the diagram?

Find some examples of good explanations. What is different about these explanations that you didn't see in the work of other students? How can you use those examples to help students develop the logic of convincing arguments for themselves?

Now look at the student work for part 2.

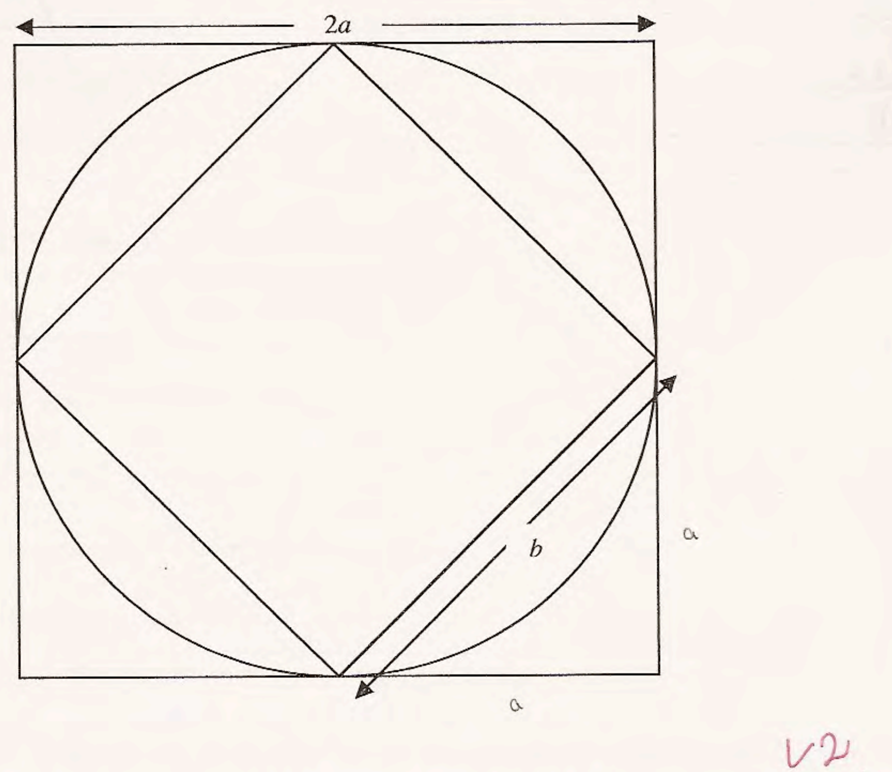
- How many of your students used Pythagorean theorem to show that $b^2 = 2a^2$?
- How many of your students used an area approach to finding b in terms of a?
- What types of errors did you see in student thinking?
- How many of your students did not attempt this part of the task?

Looking at Student Work on Circles and Squares

Student A makes a clear connection between the diagram and the algebraic expressions and shows how to find area in part 1. In part 2 the student uses the area of two triangles with a base of 2*a* and a height of *a* to connect the variables of *a* and *b*.

Student A

The diagram shows a square with a circle inside it, and a smaller square inside the circle.
The large square has sides of length 2*a* and the smaller square has sides of length *b*.



1. Explain why the diagram shows that $b^2 <$ area of the circle $< 4a^2$.

b^2 is the area of the (smaller) square $(A = b^2)$, and $4a^2$ is the area of the (larger) square $(A = (2a)^2 = 4a^2)$. We can assume from the diagram that the area of the smaller square is less than that of the circle, while the area of the circle is smaller than that of the larger square.

Student A, part 2

2. By finding b in terms of a, show that $2a^2$ < area of the circle < $4a^2$.
Show your method clearly.

√3

$\left(\frac{1}{2}(2a)\right)^2 + \left(\frac{1}{2}(2a)\right)^2 = b^2$

$a^2 + a^2 = b^2$

$2a^2 = b^2$

$2a^2$ = A of small square

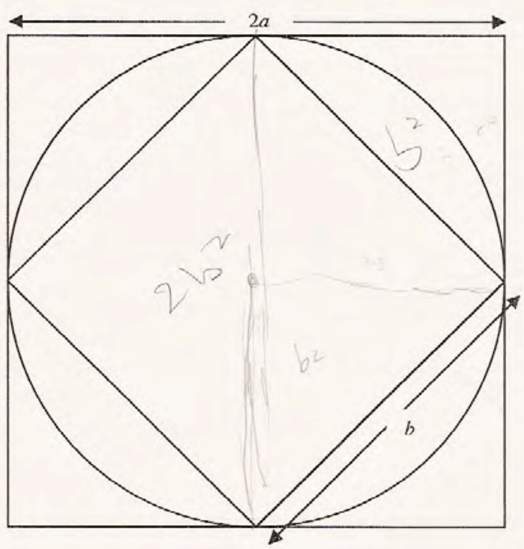
$4a^2$ = A of large square

∴ $2a^2$ < area of circle < $4a^2$

Student B uses the diagram to help think about the task. By decomposing the shape, the student connects a and b using Pythagorean theorem. The student combines quantifying the size of the small and large square with the size of the objects in the diagram to make a convincing argument in part 1 of the task.

Student B

The diagram shows a square with a circle inside it, and a smaller square inside the circle.
The large square has sides of length $2a$ and the smaller square has sides of length b.



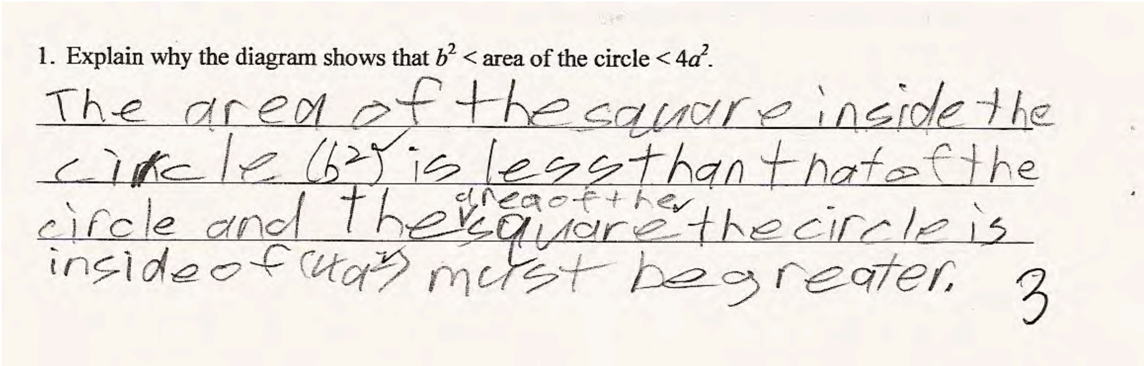
1. Explain why the diagram shows that b^2 < area of the circle < $4a^2$.

b^2 is the area of the smaller square. Since the smaller square is inside the circle, the area has to be less. and since the circle fits inside the area of the larger square ($4a^2$), the larger square has the most area.

√3

Student C shows a minimum response to part 1. The geometric representation is connected to the algebraic expressions with no reference as to how the areas were calculated.

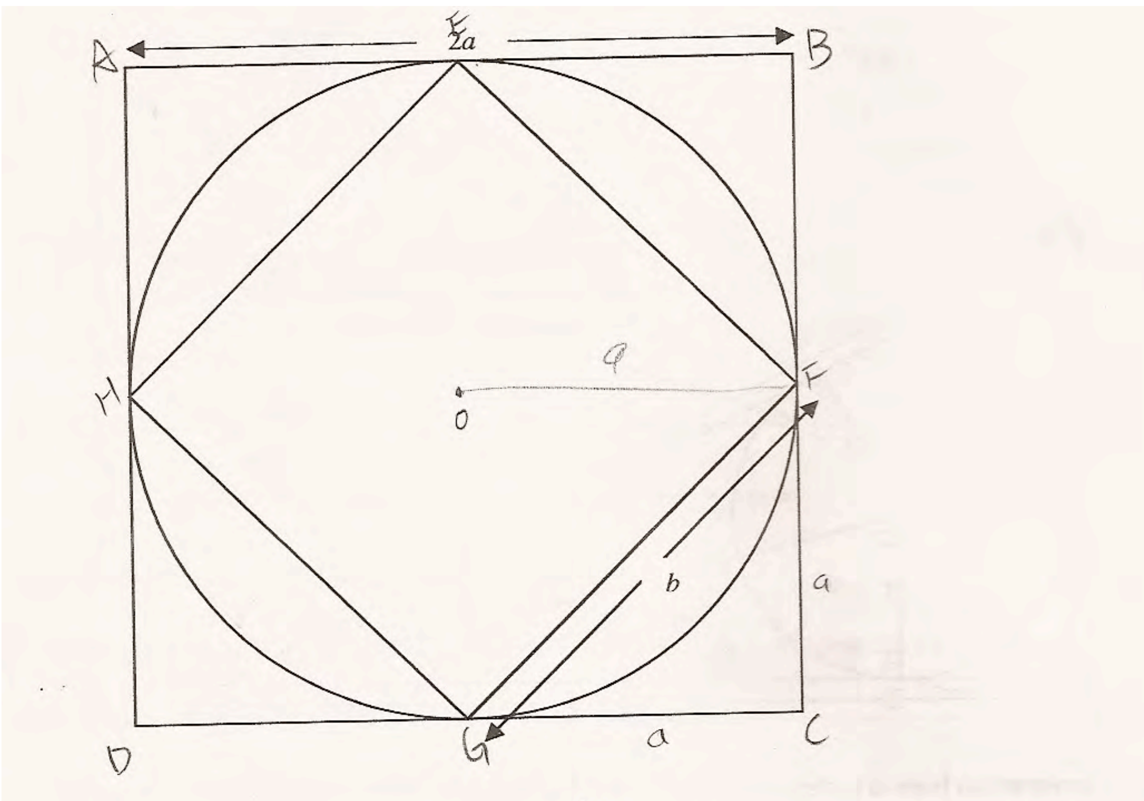
Student C


1. Explain why the diagram shows that $b^2 <$ area of the circle $< 4a^2$.

The area of the square inside the circle (b^2) is less than that of the circle and the area of the square the circle is inside of ($4a^2$) must be greater. 3

In contrast, Student D calculates the area of the circle, but doesn't connect the representations of the squares with their areas. Student D uses properties of a 45-45-right triangle to make a very convincing argument about the relationships in part 2.

Student D



. Explain why the diagram shows that $b^2 <$ area of the circle $< 4a^2$.

Because the square with side b can fit into the circle, and the circle can fit into the square with side 2a. ✓ 0 0 1

Student D, part 2

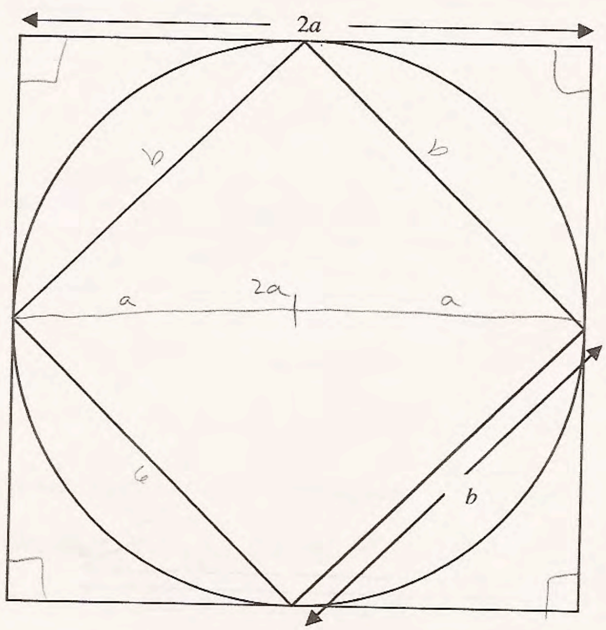
2. By finding b in terms of a, show that $2a^2 <$ area of the circle $< 4a^2$.
Show your method clearly.

$m\angle C = 90$, then $m\angle CFG = m\angle CGF = 45$, so $FC : GC : FG = a : a : b$ $= 1 : 1 : \sqrt{2}$. Then $a : b = 1 : \sqrt{2}$, $b = a\sqrt{2}$ ✓. Therefore square $EFGH = (a\sqrt{2})^2$ $= 2a^2$, and circle $O = a^2\pi$. $\pi = 3.1415926\ldots > 2$, so that $4a^2 > a^2\pi > 2a^2$ ✓. Therefore $2a^2 <$ area of the circle $< 4a^2$.

3

Look at the explanations for part 1 given by students E and F. *How would you describe their differences? Why is one given points and the other not? How do you communicate these differences in clarity to students, so that students can start to develop their own criteria for justification?*

Student E



1. Explain why the diagram shows that $b^2 <$ area of the circle $< 4a^2$.

b^2 which is the small square ✓ covers less area the the area of the circle it is in. The area of the ○ is less than $4a^2$ ✓ (the area of large □) because it is inside of it and covers less surface area.

Student F

1. Explain why the diagram shows that $b^2 <$ area of the circle $< 4a^2$.

The diagram shows this because the circle and the smaller square are inscribed in the larger square and as they are more inscribed, the sides get smaller ∧

0
0
0

Student G might understand that the area of the small square is b^2 and the area of the large square is $4a^2$, but that information is not communicated. The student basically makes the conclusion with no supporting evidence.

Student G

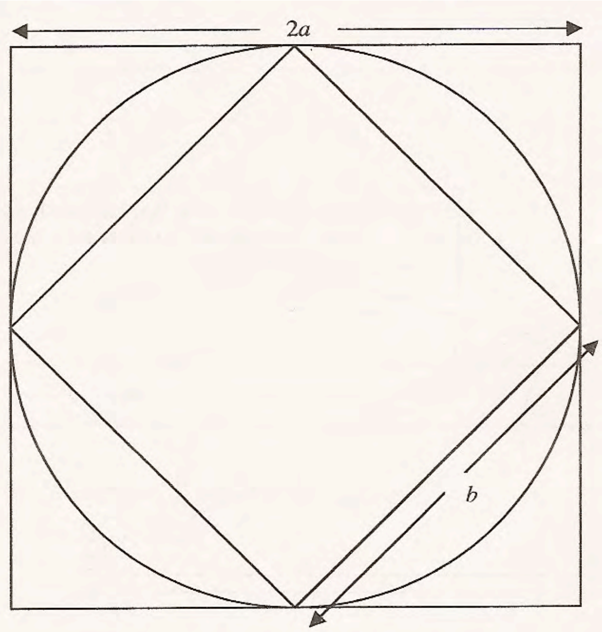
1. Explain why the diagram shows that $b^2 <$ area of the circle $< 4a^2$. bod

Since the circle is inside the square and there is empty space, it is less, ✓ and since there is empty space that the circle has, the square is less ✓

0
0
1

Student H quantifies the area of the circle, but treats all the expressions as numerical values. There is no connection to the diagram.

Student H



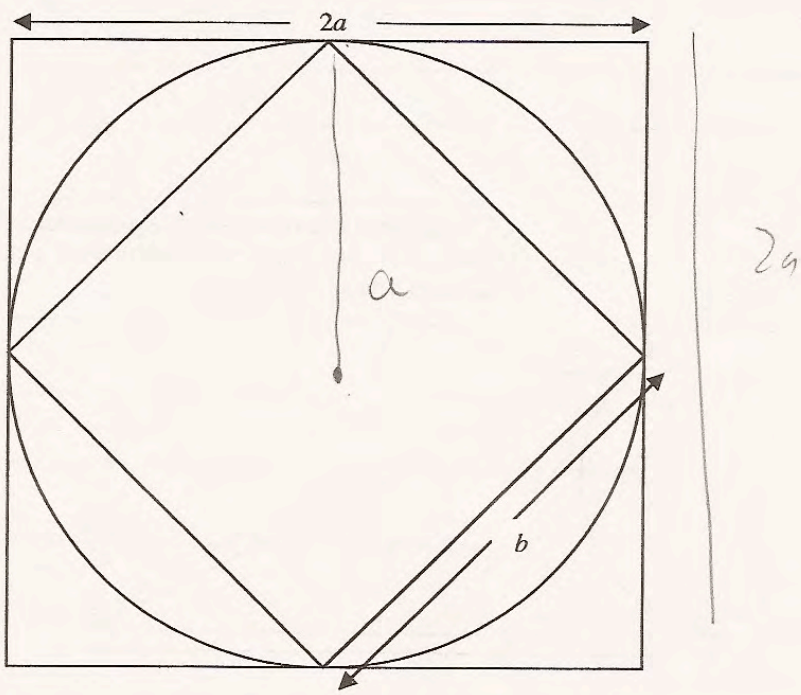
r = a

1. Explain why the diagram shows that $b^2 <$ area of the circle $< 4a^2$.

$b^2 < \pi a^2 < 4a^2$ ∧

Student I is able to quantify the area of the circle. The student assumes that b^2 is the same as $2a^2$, but doesn't make that observation explicit or give a reason why that is true. The student then simplifies by dividing all terms by a^2 to show that 2< 3.14< 4. This is a great strategy only if the connection of *a* and *b* is made.

Student I



1. Explain why the diagram shows that $b^2 <$ area of the circle $< 4a^2$.

$b^2 < \pi a^2 < 4a^2$, because it says that the square inside the circle is smaller it is a fact that $b^2 < 4a^2$. The circle has a diameter of 2a because the height of the square is 2a. and since π is 3.14 < 4 it is a fact that (πr^2) $\pi a^2 < 4a^2$... The square inside the circle is smaller because the diameter of circle is bigger there for $b^2 <$ area of circle $< 4a^2$ ✓ 0 0 1

By finding *b* in terms of *a*, show that $2a^2 <$ area of the circle $< 4a^2$.
Show your method clearly.

The ~~circle~~ areas circle is πa^2, so, 2 < 3.14 < 4. There fore $2a^2 <$ area of circle $< 4a^2$.

1

Student J also cannot explain how b is calculated.

Student J

2. By finding b in terms of a, show that $2a^2 <$ area of the circle $< 4a^2$.
Show your method clearly.

$c = \pi a^2$ $c = 3.1415a^2$ $2a^2 < 3.14a^2 < 4a^2$

$b = a\sqrt{2}$ x

Student K misses all connections of the algebraic expressions to the context of the problems and just relies on symbol manipulation.

Student K

2. By finding b in terms of a, show that $2a^2 <$ area of the circle $< 4a^2$.
Show your method clearly.

~~$b^2 < AO < 4a^2$~~ x $2a^2 \leq AO < 4a^2$

~~$\sqrt{b^2} < \sqrt{4a^2}$~~ x $2a^2 < 4a^2$

$a^2 < 2a^2$

$a < a\sqrt{2}$

$b < a\sqrt{2}$ ✓

Geometry Task 5 Circles and Squares

Student Task	Interpret a mathematical diagram. Show understanding of area. Use algebra to reason about the size of inscribed objects.
Core Idea 4 Geometry and Measurement	**Analyze characteristics and properties of two-dimensional geometric shapes, develop mathematical arguments about geometric relationships; and apply appropriate techniques, tools, and formulas to determine measurements.**

Mathematics in this task:

- Read and interpret a diagram
- Connect geometric concepts and measures with algebraic notation
- Reason about a diagram of inscribed shapes to prove why some are smaller than others
- Use area or Pythagorean theorem to connect the geometric ideas in a diagram to different algebraic expressions (using different variables) for the area of the smaller square

Areas of difficulty for geometry students:

- Reading a diagram and using it as a tool for solving a problem (less than 25% of the students made any marks on their diagram, yet all students marking their diagram scored 3 or above in the sample papers)
- Making explicit connections between a geometric representation and its algebraic notation
- Quantifying sizes when making a convincing argument
- Making assumptions about equivalencies, rather than using given information to prove relationships

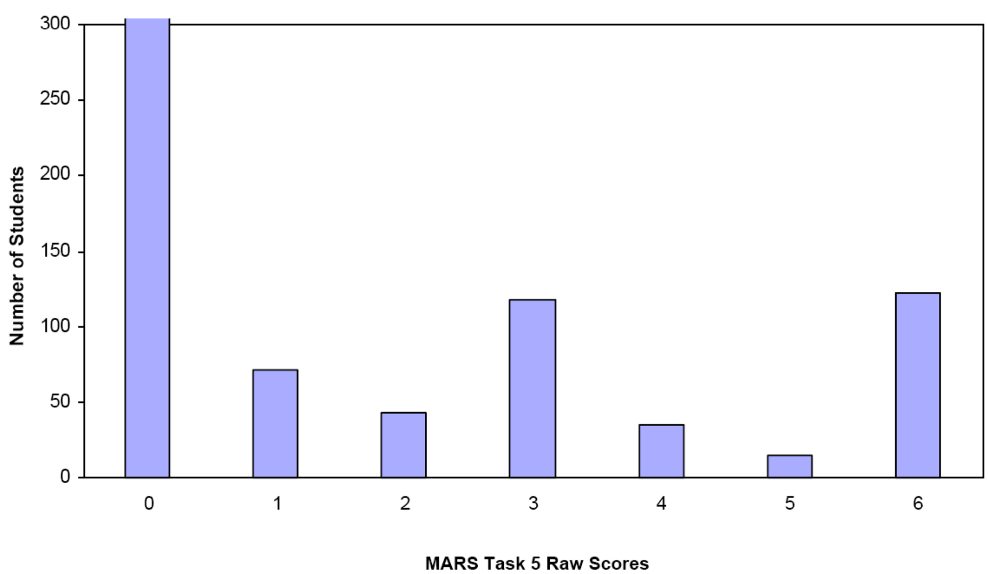
Task 5 - Circles and Squares

Mean: 1.94 StdDev: 2.25

Table 54: Frequency Distribution of MARS Test Task 5, Course 2

Task 5 Scores	Student Count	% at or below	% at or above
0	346	46.1%	100.0%
1	71	55.6%	53.9%
2	43	61.3%	44.4%
3	118	77.1%	38.7%
4	35	81.7%	22.9%
5	15	83.7%	18.3%
6	122	100.0%	16.3%

Figure 63: Bar Graph of MARS Test Task 5 Raw Scores, Course 2



The maximum score available for this task is 6 points.
The minimum score needed for a level 3 response is 3 points.

About half the students, 54%, attempted an argument about the areas of squares and a circle that concluded that the area of the circle is between the area of the small square and the large square. 39% could make a convincing argument about why the area of the circle was between the large and small square by connected the algebraic measurements in the diagrams with the calculations for areas. 16% of the students could meet all the demands of the task including using Pythagorean theorem or algebraic analysis of areas to show why the area is between $2a^2$ and $4a^2$ by explaining how a and b are related algebraically. 46% of the students scored no points on this task. 1/3 of the students with this score attempted the task.

Circles and Squares

Points	Understandings	Misunderstandings
0	34% of the students with this score attempted the task.	Many students just repeated the prompt or part of the given information. Students did not think about the connection between the objects and the algebraic notation. Many students ignored the idea that the variables were different and talked about 2 being less than 4.
1	Students could state the conclusion of the argument, but didn't give supporting evidence.	Students did not make the connection between the geometric objects and their algebraic representations.
3	Students could explain that b^2 equals the area of the small square and $4a^2$ is the area of the large square. Students could then compare the areas to show the relative sizes of the three shapes	Most students did not attempt the next part of the task. Those who did could not explain how *b* related to *a* or gave a value for *b* with no explanation of how it was calculated or derived.
6	Students could link geometric and algebraic representations to make convincing arguments about relative sizes of inscribed objects. Students were also able to connect the area or side of an object by showing the connection between two different variables.	

Implications for Instruction

Students at this level should be comfortable reading and interpreting diagrams and using them as tools to solve problems. They should routinely solve problems that involve marking up diagrams to label parts and which require adding lines to help them think about the relationships involved. Too often textbooks provide all the information, clearly labeled. While this may help students practice a particular procedure, the thinking and mathematics involved are quite different from the ability to decide what the needed measurements are and then derive them from the given information.

Students should also have frequent opportunities to make justifications. They need to hear the justifications of others and discuss what it is and is not convincing about a chain of reasoning. Students need opportunities to discuss whether enough information was provided to make a complete justification and what justification needs to be added to make the argument complete. Seeing solutions is different from experiences that help students develop the logic of making an argument or understanding the criteria for judging the quality of an argument. By looking at different examples, students can compare and contrast them. This process of compare/contrast provides a need to analyze the qualities of the argument and make them explicit.

Reflecting on the Results for Geometry as a Whole

Think about student work through the collection of tasks and the implications for instruction. What are some of the big misconceptions or difficulties that really hit home for you?

If you were to describe one or two big ideas to take away and use for planning for next year, what would they be?

What were some of the qualities that you saw in good work or strategies used by good students that you would like to help other students develop?

Four areas that stood out for the Collaborative as whole in the area of geometry were:

1. The need to <u>reinforce algebraic skills</u> throughout the year and students' difficulties working with exponents.
2. The need to help students <u>develop the logic of making a convincing argument</u>: what elements need to be made explicit, why information needs to be derived rather than assumed, what is needed to complete the logic of an argument.
3. The need to help students <u>use diagrams as tools</u> for solving problems and understand that lines can be added logically to the diagram to help derive new information necessary to solving the problem.
4. Because textbooks leave geometry until the end of the book in earlier grades, many <u>students come to geometry with little or no background skills</u>. What types of experiences needed to be front-loaded at the beginning of the course, including development of their van Hiele levels of seeing geometric attributes and properties to basic knowledge of area, perimeter, and volume and how these before students are exposed to the rigors of a high school course?

Action Research 1- What makes a convincing argument?
Students need opportunities to develop the logic of making a good justification. They need to look at work carefully to compare and contrast the qualities of a justification. Pick some sample work from a problem like part 1 of Circles and Squares, where the mathematics is easily within their reach (maybe Students A, C, D, E, F, and G). Ask students to work in pairs to compare and contrast the explanations. Having the need to use mathematical language to explain ideas to others helps students to develop their mathematical vocabulary and to become clearer about mathematical ideas. Working in pairs gives them a better opportunity to share in the responsibility of working the task. To help them get started on this type of assignment, it might be useful to give them a set of questions to ask themselves or have the class devise a set of questions. These might include:

- Where did this fact or measurement come from?
- How did the student know this fact? Where does it say that this is true?

You might ask students to write notes or make a presentation about what is missing from explanations.

As a follow up activity, at a later time, you might want to give students partially completed arguments from a different task and see if they can complete the arguments. Discussion so focus not just on the solution, but also about why they thought certain elements were needed to make the case more convincing and what they didn't like about the original argument. Again, you might start this process with some of the work from the tool kit, such as using examples from part one of Circles in Triangles.

Finally, find an interesting task, such as part 3 and 4 from Hopewell Geometry or part 2 and 3 from Rhombuses (Balanced Assessment 2006), requiring explanation and use of diagrams to see how student thinking has improved. How has this experience helped student thinking? What differences do you see in student thinking? How can this process be improved?

Action Research 2- Adding Lines to Diagrams
Students at this grade level need practice working with diagrams and see the usefulness of adding lines as a tool for solving a problem. This would be a good opportunity to meet with colleagues to think about how students learn this strategy over time.
Idea One – Look at the Video, "Can You Find the Area?" (available through the Catherine Lewis Lesson Study Website). In Day 2 and Day 3 of this 3-day lesson study on area the teacher presents carefully designed tasks to students, where adding lines is necessary to the solution of the task. This is a lesson designed for fourth and fifth graders, but gives some important insights into how students start to develop an understanding of this strategy.

Next watch the TIMMS Videos on the Japanese 8^{th} Grade Geometry Lesson. Here again, students are presented with a carefully designed problem where adding lines to a diagram is central the problem solution. Discuss with colleagues the elements of the lesson design that made the lesson so effective. How might you use these tasks with your students? What are some similar problems that would be good to help students develop this skill?

Action Research 4 –
Work with colleagues to think about how to restructure the beginning of the school year to help focus students on expectations, develop van Hiele levels, and refresh or work on some basic geometric skills (the problem from Fostering Algebraic Thinking. What does it take to "get" students ready for geometry? What activities need to be incorporated as a regular part of the program to further develop these skills?